Random Walks on Lattices

Graphs as Electrical Networks

To model a random walk on a graph, we treated the graph as an electrical network, where the edges were resistors. We then defined

- r_{xy}: the resistance between vertices x, y (note that $r_{xy} = r_{yx}$)
- $c_{xy} = \frac{1}{r_{xy}}$: the conductance is the inverse of the resistance
- $c_x = \sum_y c_{xy}$: the conductance at x
- $p_{xy} = \frac{c_{xy}}{c_x}$: the probability that a walk at x travels to y (note that $\sum_y p_{xy} = 1$)

Suppose a random walk starts at vertex a, and let vertex b be an "escape" vertex. This random walk will either return to a without going through b or will walk to b and escape. We proved that the escape probability was $\frac{c_{eff}}{c_a}$, where c_{eff} is the effective resistance between vertices a, b.

Tricks of the Trade

When calculating the escape probability on a graph, if our graph has multiple escape vertices, we can them collapse those vertices into one single escape vertex.

Also, the following theorem was stated but not proved about electrical networks: if you raise or lower the resistance of an edge in the network, you will respectively raise or lower the effective resistance between two points in the network.

For complex electrical networks, we can take advantage of this result to simply our network. When we raise the resistance of edges, the effective resistance in our modified graph is an upper bound on the effective resistance of – and a lower bound on the effective conductance of – the original graph. When we lower the resistance of edges, the effective resistance in our modified graph is a lower bound on the effective resistance of – and an upper bound of the effective conductance of – the original graph.

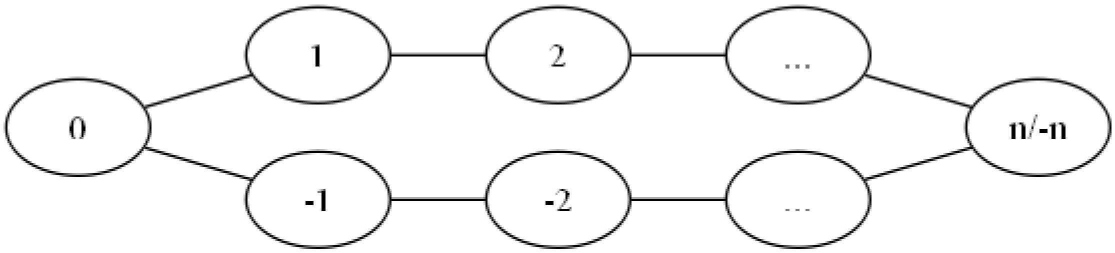
This trick has two important uses. We can lower the resistance of an edge to zero, effectively collapsing those two vertices into a single vertex, since electricity can freely flow between them. Likewise, we can raise the resistance between two vertices to infinity, effectively deleting the edge from the graph.

Escaping from a Lattice

1D Lattice

Consider a 1D lattice. We will label the start vertex as 0. Vertices to the right of 0 will be labeled 1, 2, …, and vertices to the left of 0 will be labeled -1, -2, …

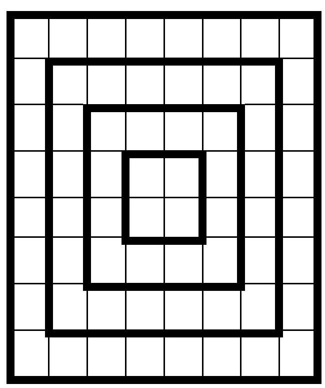
We want to model the probability a random walk starting at 0 escapes in either direction to n or $-n$. We will then take the limit as n goes to infinity to calculate the escape probability on this lattice. To do so, we collapse the vertices $n, -n$ to a single vertex.



Two paths exist to the escape vertex, each with an effective resistance of n. Thus, the effective resistance of both paths together is $\frac{n}{2}$, and the effective conductance is $\lim_{n\to\infty} \frac{2}{n} = 0$. In this graph, $c_0 = 2$. Thus, the escape probability is $p_{escape} = \frac{c_{eff}}{c_0} = \frac{0}{2} = 0$.

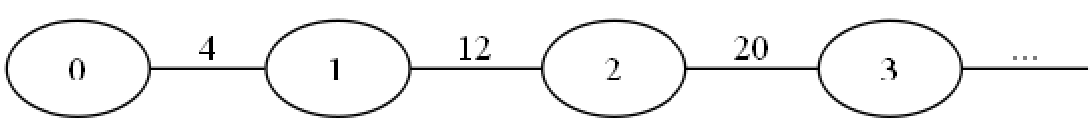
2D Lattice

For a 2D lattice, draw squares around the origin. The first square will have length 2, surrounding the origin so that it is one step away from each side of the square. The second square will have length 4, surrounding the first square so that it takes one step to travel between the first square and the second. Continue drawing squares in this fashion…



4 edges exist between the origin and the first square. The number of edges between the next two squares increases by 8 each time: there are 12 edges between the first and second squares, 20 edges between edges the second and third squares, and so on.

If we lower the resistance on the perimeter of the squares to zero, then each square collapses to a single vertex, and our 2D lattice collapses to a 1D lattice with multiple edges between vertices.



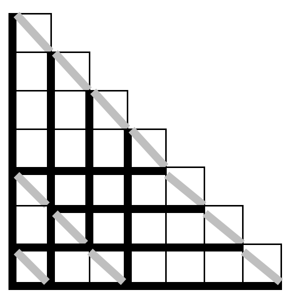
The effective resistance is $\frac{1}{4}+\frac{1}{12}+\frac{1}{20}+\cdots=\frac{1}{4}(1+\frac{1}{3}+\frac{1}{5}+\cdots)$. The infinite series $1+\frac{1}{3}+\frac{1}{5}+\cdots$ does not converge, so the effective resistance is infinite and the effective conductance is 0 on the modified graph. Since we lowered the resistance, the effective conductance is an upper bound, meaning that the effective conductance on the original graph is also 0. Finally, $c_{orig\,in}=4$, so the escape probability is $p_{escape}=\frac{c_{eff}}{c_{origin}}=\frac{0}{4}$.

3D Lattice

To find an upper bound, let us repeat the same trick we used for the 2D lattice, except that we will be drawing cubes instead of squares. If we define cube 0 as the origin, then there are $6(2n+1)^2$ edges between the n^{th} cube and the $(n+1)^{\text{st}}$ cube.

The effective resistance becomes $\frac{1}{6}(1+\frac{1}{9}+\frac{1}{25}+\cdots)$, but the infinite sequence in this case will converge. Since this formula is already a lower bound, we can lower it further by taking the first couple of terms and throwing out the remaining terms. The effective resistance is at least $\frac{1}{5}$ and the effective conductance is at most 5. For three dimensions, we now have $c_{origin}=6$, thus the escape probability is $p_{escape}\leq\frac{c_{eff}}{c_{origin}}=\frac{5}{6}$.

To find a lower bound in two dimensions, define a series of lines by the equation $x+y=2^n-1$. Each time the walk hits the line, draw one path in the x direction and another path in the y direction to the next line. For edges not on the paths, raise the resistance to infinity. As you can see, these paths branch out in two directions like a tree, where the line at $x+y=2^n-1$ represents the n^{th} level of the tree. The effective resistance to the next level of the tree starts at 1 from the root and doubles each time.



In three dimensions, with the line $x+y+z=2^n-1$, a similar concept happens, except that each vertex branches out in 3 directions, not 2. The effective resistance is $\frac{1}{3}(1)+\frac{1}{9}(2)+\frac{1}{27}(4)+\cdots=\frac{1}{3}\left(1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\cdots\right)=\frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right)=\frac{1}{3}(3)=1$. The escape probability is $p_{escape}\geq\frac{c_{eff}}{c_{origin}}=\frac{1}{6}$.

Combining the two bounds, we have $\frac{1}{6}\leq p_{escape}\leq\frac{5}{6}$.